10th Benelux Mathematical Olympiad

Luxembourg, 27th–29th April 2018

Language: **English**

The problems are not ordered by estimated difficulty.

Problem 1. (a) Determine the minimal value of

$$\left(x+\frac{1}{y}\right)\left(x+\frac{1}{y}-2018\right)+\left(y+\frac{1}{x}\right)\left(y+\frac{1}{x}-2018\right),$$

where x and y vary over the positive reals.

(b) Determine the minimal value of

$$\left(x+\frac{1}{y}\right)\left(x+\frac{1}{y}+2018\right)+\left(y+\frac{1}{x}\right)\left(y+\frac{1}{x}+2018\right),$$

where x and y vary over the positive reals.

Problem 2. In the land of Heptanomisma, four different coins and three different banknotes are used, and their denominations are seven different (non-zero) natural numbers. The denomination of the smallest banknote is greater than the sum of the denominations of the four different coins. A tourist has exactly one coin of each denomination and exactly one banknote of each denomination, but he cannot afford the book on numismatics he wishes to buy. However, the mathematically inclined shopkeeper offers to sell the book to the tourist at a price of his choosing, provided that he can pay this price in more than one way.
(*The tourist can pay a price in more than one way if there are two different subsets of his coins and notes, the denominations of which both add up to this price.*)

(a) Prove that the tourist can purchase the book if the denomination of each banknote is smaller than 49.

(b) Show that the tourist may have to leave the shop empty-handed if the denomination of the largest banknote is 49.

Problem 3. Let ABC be a triangle with orthocentre H, and let D, E, and F denote the respective midpoints of line segments AB, AC, and AH. The reflections of B and C in F are P and Q, respectively.

(a) Show that lines PE and QD intersect on the circumcircle of triangle ABC.

(b) Prove that lines PD and QE intersect on line segment AH.

Problem 4. An integer $n \geqslant 2$ having exactly s positive divisors $1 = d_1 < d_2 < \cdots < d_s = n$ is said to be *good* if there exists an integer k, with $2 \leqslant k \leqslant s$, such that $d_k > 1 + d_1 + \cdots + d_{k-1}$. An integer $n \geqslant 2$ is said to be *bad* if it is not good.

(a) Show that there are infinitely many bad integers.

(b) Prove that, among any seven consecutive integers all greater than 2, there are always at least four good integers.

(c) Show that there are infinitely many sequences of seven consecutive good integers.

Time: 4 hours and 30 minutes.
Each problem is worth 7 points.